

# Relations between the convexity of a set and the differentiability of its support function

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## Abstract

It is known that, in finite dimensions, the support function of a compact convex set with non empty interior is differentiable excepting the origin if and only if the set is strictly convex. In this paper we realize a thorough study of the relations between the differentiability of the support function on the interior of its domain and the convexity of the set, mainly for unbounded sets. Then we revisit some results related to the differentiability of the cost function associated to a production function.

## 1 Introduction

The celebrated Shephard lemma, which is considered to be “a major result in microeconomics having applications in the theory of the firm and in consumer choice” (see [http://en.wikipedia.org/wiki/Shephard's\\_lemma](http://en.wikipedia.org/wiki/Shephard's_lemma)) is related to the differentiability of the cost function in economics. The cost function is defined by

$$g : \mathbb{R}_{++}^p \rightarrow \mathbb{R}, \quad g(x) := \inf \{ \langle x, a \rangle \mid a \in A \},$$

where  $A$  is a nonempty subset of  $\mathbb{R}_+^p$ . More precisely,  $A = \{u \in \mathbb{R}_+^p \mid f(u) \geq y\}$ , where  $f : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$  is a production function.

Clearly, the function  $g$  above is strongly related to the support function which is defined as

$$\sigma_A : X^* \rightarrow \overline{\mathbb{R}}, \quad \sigma_A(x^*) := \sup \{ \langle x^*, u \rangle \mid u \in A \},$$

where  $X$  is a (finite dimensional) real normed space whose topological dual is denoted by  $X^*$ , and  $A \subset X$  is a nonempty set.

Because  $g(x^*) = -\sigma_A(-x^*)$  for  $x^* \in \mathbb{R}^p$  (where  $X = \mathbb{R}^p$  is endowed with the usual Euclidean norm), any property of the support function  $\sigma_A$  can be translated into a corresponding property of the cost function  $g$ .

In the economics literature one can find several results related to Shephard's lemma and to the differentiability of the cost function; see [3], [4], [5], [6], [8], [11], [12]. Our aim is to study the connection between the differentiability of the support function  $\sigma_A$  and the convexity of  $A$ , and to revisit some results related to Shephard's lemma.

The rest of the paper is organized as follows. In Section 2 we present preliminary notions and results. In Section 3 we recall several results from [16] concerning the differentiability of  $\sigma_A$  for  $A$  convex. Section 4 contains the main results of this paper. We present several

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conditions on the set  $A$  which imply the differentiability of  $\sigma_A$  (on certain sets) and conditions on  $A$  under which the differentiability of  $\sigma_A$  on  $\text{int}(\text{dom } \sigma_A)$  implies the convexity of  $A$ . Moreover, we associate to a set  $A$  satisfying condition (H) a function  $F_A$  and establish relationships between properties of the set  $A$  and properties of  $F_A$ . Then, in Section 5, we apply the results in Section 4 to the problem of the differentiability of the cost function and discuss several results on Shephard's lemma from the economics literature.

## 2 Preliminaries

In the following we assume that  $X$  is a nontrivial real finite dimensional normed space whose dual is denoted by  $X^*$ . We identify  $(X^*)^*$  with  $X$ . However, the reader can take  $X$  an Euclidean space and identify  $X^*$  with  $X$ . For  $A \subset X$  we denote by  $\text{aff } A$ ,  $\text{lin}_0 A$ ,  $\text{cl } A$ ,  $\text{int } A$ ,  $\text{rint } A$ ,  $\text{bd } A$ ,  $\text{rbd } A$ ,  $\text{conv } A$ ,  $\overline{\text{conv}} A$  the affine hull of  $A$ , the linear space parallel to  $\text{aff } A$ , the closure of  $A$ , the interior of  $A$ , the relative interior of  $A$  (that is the interior of  $A$  w.r.t.  $\text{aff } A$ ), the boundary of  $A$  (hence  $\text{bd } A = \text{cl } A \setminus \text{int } A$ ), the relative boundary of  $A$  (hence  $\text{rbd } A = \text{cl } A \setminus \text{rint } A$ ), the convex hull of  $A$  and  $\text{cl}(\text{conv } A)$ , respectively. When  $A \subset B \subset X$ , we write  $\text{int}_B A$  and  $\text{bd}_B A$  for the interior and boundary of  $A$  as subset of  $B$  endowed with the induced topology. Recall that the recession cone of  $A \neq \emptyset$  is the set

$$A_\infty := \{u \in X \mid \exists (t_n) \subset (0, \infty), t_n \rightarrow 0, \exists (a_n) \subset A : t_n a_n \rightarrow u\}.$$

Clearly,  $A_\infty$  is a (closed) cone; in particular,  $0 \in A_\infty$ . We have that  $A_\infty = \{0\}$  if and only if  $A$  is bounded. When  $A$  is closed and convex we have that  $A_\infty = \cap_{t>0} t(A - a)$ , where  $a \in A$ . In this case  $A_\infty$  is also convex; moreover,  $A_\infty$  is pointed, that is,  $A_\infty \cap (-A_\infty) = \{0\}$ , if and only if  $A$  does not contain any line.

For the the set  $A \subset X$  we set

$$A^+ := \{x^* \in X^* \mid \langle x, x^* \rangle \geq 0 \ \forall x \in A\}, \quad A^- := -A^+, \quad A^\perp := A^+ \cap A^-,$$

where  $\langle x, x^* \rangle := x^*(x)$  for  $x \in X$  and  $x^* \in X^*$ .

For  $P \subset X$  a closed convex cone we set

$$P^\# := \{x^* \in X^* \mid \langle x, x^* \rangle > 0 \ \forall x \in P \setminus \{0\}\}.$$

It is known that  $P^\# \neq \emptyset$  if and only if  $P$  is pointed (that is  $P \cap (-P) = \{0\}$ ); note that  $\{0\}^\# = X^*$ . Furthermore,  $P^\# = \text{int } P^+$ . Moreover, for  $x, x' \in X$  we write  $x \geq_P x'$  for  $x - x' \in P$ ,  $x \geq_P x'$  for  $x - x' \in P \setminus \{0\}$  and  $x >_P x'$  for  $x - x' \in \text{int } P$ .

Recall that the domain  $\text{dom } f$  of the function  $f : X \rightarrow \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$  is the set  $\{x \in X \mid f(x) < \infty\}$ , while the epigraph of  $f$  is the set  $\text{epi } f := \{(x, \lambda) \in X \times \mathbb{R} \mid f(x) \leq \lambda\}$ ;  $f$  is proper if  $\text{dom } f \neq \emptyset$  and  $f(x) \neq -\infty$  for every  $x \in X$ . If  $f$  is proper, its recession function is  $f_\infty : X \rightarrow \overline{\mathbb{R}}$  whose epigraph is  $(\text{epi } f)_\infty$ ;  $f_\infty$  is lower semicontinuous (lsc for short) and positively homogeneous, that is  $f_\infty(tu) = tf_\infty(u)$  for all  $u \in X$  and  $t \in \mathbb{P} := (0, \infty)$ . The conjugate of  $f$  is the function

$$f^* : X^* \rightarrow \overline{\mathbb{R}}, \quad f^*(x^*) := \sup \{\langle x, x^* \rangle - f(x) \mid x \in X\}$$

and the subdifferential of the proper function  $f$  at  $x \in \text{dom } f$  is

$$\partial f(x) := \{x^* \in X^* \mid \langle x' - x, x^* \rangle \leq f(x') - f(x) \ \forall x' \in X\}$$

and  $\partial f(x) = \emptyset$  for  $x \in X \setminus \text{dom } f$ . Of course, the domain of  $\partial f$  is  $\text{dom } \partial f := \{x \in X \mid \partial f(x) \neq \emptyset\} (\subset \text{dom } f)$ . Clearly,  $\sigma_A = (\iota_A)^*$ , where the indicator function  $\iota_A$  of  $A \subset X$  is defined by  $\iota_A(x) := 0$  for  $x \in A$  and  $\iota_A(x) := +\infty$  for  $x \in X \setminus A$ .

Coming back to the support function, it is well known that for the nonempty set  $A \subset X$  we have that  $\sigma_A = \sigma_{\text{conv } A} = \sigma_{\text{cl } A}$ , which shows that (in many problems) we can assume that  $A$  is a (nonempty) closed convex set.

For  $\emptyset \neq C = \overline{\text{conv}} C$  (that is  $C$  is a nonempty closed convex set) we have that

$$(C_\infty)^- = \text{cl}(\text{dom } \sigma_C),$$

whence

$$\text{int}(\text{dom } \sigma_C) = \text{int} (C_\infty)^-. \quad (1)$$

Hence  $\text{int}(\text{dom } \sigma_C) \neq \emptyset$  if and only if  $C_\infty$  is a pointed cone. Moreover,

$$\text{lin}_0 C = X \Leftrightarrow (\text{lin}_0 C)^\perp = \{0\} \Leftrightarrow \text{int } C \neq \emptyset,$$

and  $\partial \sigma_C(x^*) = C$  for every  $x^* \in (\text{lin}_0 C)^\perp$ , whence  $(\text{lin}_0 C)^\perp \subset \text{dom } \partial \sigma_C$ . It follows that  $\sigma_C$  is differentiable at  $x^* \in (\text{lin}_0 C)^\perp$  iff  $C$  is a singleton (in which case  $(\text{lin}_0 C)^\perp = \text{dom } \sigma_C = X^*$  and  $\sigma_C$  is differentiable). Furthermore,

$$(\text{lin}_0 C)^\perp = \text{dom } \partial \sigma_C \Leftrightarrow (\text{lin}_0 C)^\perp = \text{dom } \sigma_C \Leftrightarrow C = \text{aff } C.$$

If  $C$  is unbounded (or, equivalently,  $C$  is not compact) then  $(\text{lin}_0 C)^\perp \cap \text{int}(\text{dom } \sigma_C) = \emptyset$ .

In [16, Prop. 1] it is shown that for the nonempty closed convex set  $C \subset X$  one has that  $C_\infty$  is pointed iff there are  $\bar{x} \in X$  and a closed convex pointed cone  $P$  such that  $C \subset \bar{x} + P$ . From this we get immediately that for the nonempty set  $A \subset X$  one has that  $\text{int}(\text{dom } \sigma_A) \neq \emptyset$  iff there exist  $\bar{x} \in X$  and a closed convex pointed cone  $P$  such that  $A \subset \bar{x} + P$ . Because  $\text{dom } \sigma_{A+x} = \text{dom } \sigma_A$  for every  $x \in X$ , we may (and we shall often do) assume that  $A \subset P$ .

We have that

$$\partial \sigma_A(0) = \overline{\text{conv}} A, \quad \partial \sigma_A(x^*) = \{u \in \overline{\text{conv}} A \mid \langle u, x^* \rangle = \sigma_A(x^*)\} \quad (2)$$

for every  $x^* \in X^*$ .

Because  $\sigma_A$  is a sublinear (hence convex) function,  $\sigma_A$  is locally Lipschitz on the interior of its domain, and so its Gâteaux and Fréchet differentiability coincide. This is the reason for speaking simply about the differentiability of  $\sigma_A$  in the sequel.

Theorem 25.1 in [10] states that the proper convex function  $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$  is differentiable at  $\bar{x} \in \text{dom } f$  if and only if  $\partial f(\bar{x})$  is a singleton, in which case  $\partial f(\bar{x}) = \{\nabla f(\bar{x})\}$ . Hence  $\sigma_A$  is differentiable at  $\bar{x}^*$  if and only if  $\partial \sigma_A(\bar{x}^*)$  is a singleton. Using (2) it follows that  $\sigma_A$  is differentiable at 0 if and only if  $A$  is a singleton, in which case  $\sigma_A$  is a linear functional.

In the next section we recall some results related to the differentiability of  $\sigma_A$  in the case  $A$  is a closed convex set with  $\text{int}(\text{dom } \sigma_A) \neq \emptyset$ . These results suggest the kind of conditions to be imposed in order that the differentiability of  $\sigma_A$  imply the convexity of  $A$ .

### 3 The convex case

Let  $C \subset X$  be a nonempty closed convex set with  $\text{int}(\text{dom } \sigma_C) \neq \emptyset$ , or equivalently, there exist  $\bar{x} \in X$  and a pointed closed convex cone  $P \subset X$  with  $C \subset \bar{x} + P$ . We recall some results concerning the differentiability of  $\sigma_C$  which can be found in [16].

**Theorem 1** ([16, Thm. 1])  $\sigma_C$  is differentiable on  $\text{dom } \partial\sigma_C \setminus (\text{lin}_0 C)^\perp$  if and only if

$$\forall x, x' \in C, \ x \neq x', \ \forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)x' \in \text{rint } C, \quad (3)$$

or, equivalently,

$$\forall x, x' \in \text{rbd } C, \ x \neq x', \ \forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)x' \notin \text{rbd } C. \quad (4)$$

Recall that a closed convex set  $C$  with nonempty interior (hence  $\text{rint } C = \text{int } C$ ) is called strictly convex if condition (3) is verified.

In the case in which  $C$  is compact the following result holds.

**Corollary 2** ([16, Cor. 4]) Assume that  $C$  is compact. Then

- (i)  $\sigma_C$  is differentiable on  $X^* \setminus (\text{lin}_0 C)^\perp$  if and only if  $C$  verifies condition (3).
- (ii)  $\sigma_C$  is differentiable on  $X^* \setminus \{0\}$  if and only if either  $C$  is a singleton or  $\text{int } C \neq \emptyset$  and  $C$  is strictly convex.

In the case in which  $C$  is not compact one has the next result.

**Theorem 3** ([16, Thm. 6]) Assume that  $C$  is unbounded. Then  $\sigma_C$  is differentiable on  $\text{int}(\text{dom } \sigma_C)$  if and only if

$$\forall x, x' \in S_C, \ x \neq x', \ \forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)x' \notin S_C, \quad (5)$$

where

$$S_C := \partial\sigma_C(\text{int}(\text{dom } \sigma_C)).$$

One has also the following result.

**Proposition 4** ([16, Prop. 7]) Assume that  $C$  is unbounded. If

$$\forall x, x' \in E_C, \ x \neq x', \ \forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)x' \notin E_C, \quad (6)$$

where

$$E_C := C \setminus [C + (C_\infty \setminus \{0\})],$$

or, equivalently,

$$\forall x, x' \in C, \ x \neq x', \ \forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)x' \in C + (C_\infty \setminus \{0\}), \quad (7)$$

then  $\sigma_C$  is differentiable on  $\text{int}(\text{dom } \sigma_C)$ . Moreover, if  $\dim(\text{lin}_0 C) \leq 2$  then the converse is also true.

**Remark 1** In [16, Lem. 5] it is shown that for  $C$  unbounded one has  $S_C \subset E_C \subset \text{rbd } C$ , and so (4)  $\Rightarrow$  (6)  $\Leftrightarrow$  (7)  $\Rightarrow$  (5). In fact, taking into account that  $\text{int}(\text{dom } \sigma_C) \subset \text{dom } \partial\sigma_C$ , if  $C$  is unbounded and  $\sigma_C$  is differentiable on  $\text{dom } \partial\sigma_C \setminus (\text{lin}_0 C)^\perp$ , then  $\text{dom } \partial\sigma_C \setminus (\text{lin}_0 C)^\perp = \text{int}(\text{dom } \sigma_C)$ .

In [16] it is shown that the set  $A$  defined in (20) is a closed convex set included in  $\mathbb{R}_+^3$  with  $A_\infty = \mathbb{R}_+^3$  for which  $\sigma_A$  is differentiable on  $\text{int}(\text{dom } \sigma_A)$  but for which (7) does not hold ([16, Prop. 10]). Hence, in general, (7) does not imply (4).

**Theorem 5** ([16, Thm. 12]) *Let  $K \subset X$  be a pointed closed convex cone and let  $A \subset X$  be a nonempty closed convex set such that  $A_\infty \subset K$ . Then  $\sigma_A$  is differentiable on  $-K^\#$  if and only if*

$$\forall x, x' \in SE(A; K), \quad x \neq x', \quad \forall \lambda \in (0, 1) : \lambda x + (1 - \lambda)x' \notin SE(A; K), \quad (8)$$

where

$$SE(A; K) := \cup_{x^* \in K^\#} \partial \sigma_A(-x^*) = \partial \sigma_A(-K^\#).$$

Note that Theorem 3 can be obtained from Theorem 5 taking  $K := A_\infty$  because  $S_A = SE(A; A_\infty)$  and  $K^\# = -\text{int}(\text{dom } \sigma_A)$ .

## 4 Relations between the differentiability of $\sigma_A$ and the convexity of $A$

Throughout this section  $\mathbb{N} \ni p := \dim X \geq 2$  (the case  $\dim X = 1$  being trivial) and  $A \subset X$  is a nonempty closed set with the property that  $P := (\overline{\text{conv}} A)_\infty$  is pointed. We consider the multifunction

$$W_A : X^* \rightrightarrows X, \quad W_A(x^*) := \{a \in A \mid \langle a, x^* \rangle = \sigma_A(x^*)\} = A \cap \partial \sigma_A(x^*). \quad (9)$$

Of course,  $\text{dom } W_A \subset \text{dom } \sigma_A \subset P^-$ .

**Proposition 6** *The following assertions hold:*

- (a)  $\text{int } P^- = \text{int}(\text{dom } \sigma_A) \neq \emptyset$  and  $\sigma_A$  is continuous on  $\text{int } P^-$ ;
- (b)  $W_A(x^*)$  is nonempty and compact for every  $x^* \in \text{int } P^-$ ;
- (c)  $\partial \sigma_A(x^*)$  is nonempty convex and compact for every  $x^* \in \text{int } P^-$ .

*Proof.* (a), (c) The equality  $\text{int } P^- = \text{int}(\text{dom } \sigma_A)$  was observed above. Because  $\sigma_A$  is convex we have that  $\sigma_A$  is continuous on  $\text{int}(\text{dom } \sigma_A)$  by a well-known result in Convex Analysis. Since the subdifferential of a proper convex function is nonempty, convex and compact at any point of continuity from its domain, (c) follows.

(b) Fix  $x^* \in \text{int } P^- = -P^\#$ . Let  $u_0 \in A$  be fixed and take  $A_0 := \{u \in A \mid \langle u, x^* \rangle \geq \langle u_0, x^* \rangle\}$ . Clearly  $A_0$  is nonempty and closed. Assume that  $A_0$  is not bounded. Then there exists  $(u_n) \subset A_0$  with  $\|u_n\| \rightarrow \infty$ . We may assume that  $\|u_n\|^{-1} u_n \rightarrow v$ , and so  $v \in P \setminus \{0\}$ . Since  $\langle u_n, x^* \rangle \geq \langle u_0, x^* \rangle$  for every  $n$ , dividing by  $\|u_n\|$  and passing to the limit we get the contradiction  $0 > \langle v, x^* \rangle \geq 0$ . Hence  $A_0$  is bounded, and so  $A_0$  is compact. Since  $\sigma_A(x^*) = \sup \{\langle u, x^* \rangle \mid x^* \in A_0\}$  and  $A_0$  is compact, there exists  $u \in A_0 \subset A$  with  $\langle u, x^* \rangle = \sigma_A(x^*) \in \mathbb{R}$ . It follows that  $W_A(x^*)$  is nonempty and compact.  $\square$

Note that we can have that  $W_A(x^*)$  is nonempty and compact without having  $x^* \in \text{int } P^-$ .

**Example 1** Take  $X = \mathbb{R}^2$  endowed with the Euclidean norm and

$$A := \{(a, b) \in \mathbb{R}^2 \mid b \geq |a| (1 + (a^2 + 1)^{-1})\}.$$

Then  $C := \overline{\text{conv}} A = \{(a, b) \in \mathbb{R}^2 \mid b \geq |a|\}$ . We have that  $\text{dom } \sigma_A = -C$  and  $W_A(-1, -1) = \{(0, 0)\}$  is compact and nonempty; clearly,  $(0, 0) \notin \text{int}(\text{dom } \sigma_A)$ .

However, the next result holds.

**Proposition 7** *Let  $B \subset X$  be a nonempty closed set. Then  $x^* \in \text{int}(\text{dom } \sigma_B)$  if and only if  $\partial\sigma_B(x^*)$  is nonempty and compact.*

Proof. Set  $C := \overline{\text{conv}} B$  and  $Q := C_\infty$ . If  $x^* \in \text{int}(\text{dom } \sigma_B)$  then, by (1),  $\text{int } Q^- \neq \emptyset$ , and so  $Q^-$  is pointed. From Proposition 6 (c) we have that  $\partial\sigma_B(x^*)$  is nonempty and compact.

Assume now that  $\partial\sigma_B(x^*)$  is nonempty and compact. Take  $f : X \rightarrow \overline{\mathbb{R}}$ ,  $f := -x^* + \iota_C$ ; then  $f$  is a proper lsc convex function, and  $\{x \in X \mid f(x) = \inf f\} = \partial\sigma_B(x^*)$ . Since  $\sigma_B(x^*)$  is nonempty and compact, it follows that  $0 \in \text{int}(\text{dom } f^*)$ , that is,  $x^* \in \text{int}(\text{dom } \sigma_C)$  (see e.g. [15, Exer. 2.41]).  $\square$

**Proposition 8** *One has that  $\partial\sigma_A(x^*) = \text{conv } W_A(x^*) \subset \text{conv } A$  for every  $x^* \in \text{int } P^-$ .*

Proof. Fix  $x^* \in \text{int } P^-$ . From (9) we have that  $W_A(x^*) \subset \partial\sigma_A(x^*)$ , and so  $\text{conv } W_A(x^*) \subset \partial\sigma_A(x^*)$ . Let  $u \in \partial\sigma_A(x^*)$ . Hence  $u \in \overline{\text{conv}} A$  and  $\langle x^*, u \rangle = \sigma_A(x^*)$ . Using the Carathéodory theorem, we find  $(\lambda_n^k)_{n \geq 1} \subset [0, 1]$  and  $(u_n^k)_{n \geq 1} \subset A$  for  $k \in \overline{1, p+1}$  such that  $\sum_{k=1}^{p+1} \lambda_n^k = 1$ ,  $u_n := \sum_{k=1}^{p+1} \lambda_n^k u_n^k \rightarrow u$ .

We claim that the sequences  $(\lambda_n^k u_n^k)_{n \geq 1}$  are bounded. In the contrary case we may assume that

$$\|\lambda_n^1 u_n^1\| \leq \|\lambda_n^2 u_n^2\| \leq \dots \leq \|\lambda_n^{p+1} u_n^{p+1}\| \quad \forall n \geq 1$$

and  $\|\lambda_n^{p+1} u_n^{p+1}\| \rightarrow \infty$  as  $n \rightarrow \infty$ . Taking subsequences if necessary, we may assume that  $\|\lambda_n^{p+1} u_n^{p+1}\|^{-1} \lambda_n^k u_n^k \rightarrow v_k$  for  $k \in \overline{1, p+1}$ . Since  $\lambda_n^k \in [0, 1]$  and  $\|\lambda_n^{p+1} u_n^{p+1}\| \rightarrow \infty$  we have that  $\|\lambda_n^{p+1} u_n^{p+1}\|^{-1} \lambda_n^k \rightarrow 0$ . Since  $u_n^k \in A \subset \overline{\text{conv}} A$ , we obtain that  $v_k \in (\overline{\text{conv}} A)_\infty = P$  for every  $k \in \overline{1, p+1}$ . From  $\sum_{k=1}^{p+1} \lambda_n^k u_n^k \rightarrow u$  we get  $v_1 + \dots + v_{p+1} = 0$ . Since  $P$  is pointed and  $v_{p+1} \neq 0$  we get a contradiction. Hence the sequences  $(\lambda_n^k u_n^k)_{n \geq 1}$  are bounded is true (for  $k \in \overline{1, p+1}$ ).

We may assume that  $\lambda_n^k \rightarrow \lambda^k$  and  $\lambda_n^k u_n^k \rightarrow v^k$  for every  $k \in \overline{1, p+1}$ ; moreover, we may assume that  $\lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^{p+1}$ . Since  $\sum_{k=1}^{p+1} \lambda^k = 1$ ,  $\lambda^{p+1} > 0$ . If  $\lambda^k > 0$  then  $u_n^k \rightarrow u^k := (\lambda^k)^{-1} v_k \in A$ . If  $\lambda^1 > 0$  then  $u = \sum_{k=1}^{p+1} \lambda^k u^k \in \text{conv } A$ . If  $\lambda^1 = 0$  take  $k_0 \in \overline{1, p}$  such that  $\lambda^{k_0} = 0$  and  $\lambda^{k_0+1} > 0$ . Then  $u = \bar{v} + \bar{u}$  with  $\bar{v} := \sum_{k=1}^{k_0} v^k$ ,  $\bar{u} := \sum_{k=k_0+1}^{p+1} \lambda^k u^k \in \text{conv } A$ . Since  $\lambda^k = 0$  for  $1 \leq k \leq k_0$ , we have that  $v_k \in A_\infty \subset P$  for such  $k$ , and so  $\bar{v} \in P$ . We have that

$$\sigma_A(x^*) = \langle u, x^* \rangle = \langle \bar{v}, x^* \rangle + \langle \bar{u}, x^* \rangle \leq \langle \bar{u}, x^* \rangle = \sum_{k=k_0+1}^{p+1} \lambda^k \langle u^k, x^* \rangle \leq \sigma_A(x^*).$$

Therefore,  $\langle \bar{v}, x^* \rangle = 0$ , and so  $\bar{v} = 0$  because  $x^* \in \text{int } P^+ = P^\#$ . It follows that  $u = \bar{u} \in \text{conv } A$ .  $\square$

**Proposition 9** *Let  $x^* \in \text{int } P^-$ . Then  $\sigma_A$  is differentiable at  $x^*$  if and only if  $W_A(x^*)$  is a singleton. In this case  $\nabla\sigma_A(x^*) \in A$  and  $W_A(x^*) = \{\nabla\sigma_A(x^*)\}$ .*

Proof. Because  $\sigma_A$  is convex and continuous at  $x^* \in \text{int}(\text{dom } \sigma_A)$ ,  $\sigma_A$  is differentiable at  $x^*$  if and only if  $\partial\sigma_A(x^*)$  is a singleton. Using Proposition 8, this happens exactly when  $W_A(x^*)$  is a singleton.  $\square$

**Corollary 10** *If  $W_A(x^*)$  is a singleton for every  $x^* \in \text{int } P^-$  then  $\sigma_A$  is differentiable on  $\text{int}(\text{dom } \sigma_A) = \text{int } P^-$ .*

**Corollary 11** *If  $\sigma_A$  is differentiable on  $\text{dom } \partial\sigma_A \setminus (\text{lin}_0 A)^\perp$  then  $\text{rbd}(\overline{\text{conv}} A) \subset A$ .*

Proof. Set  $C := \overline{\text{conv}} A$ ; it follows that  $\text{lin}_0 A = \text{lin}_0 C$ .

Let  $u \in \text{rbd} C$ . Using a separation theorem, there exists  $x^* \in X^* \setminus (\text{lin}_0 C)^\perp$  (that is  $x^*$  is not constant on  $C$ ) such that  $\langle u, x^* \rangle = \sigma_C(x^*) = \sigma_A(x^*)$ . Therefore,  $x^* \in \text{dom } \partial\sigma_A \setminus (\text{lin}_0 A)^\perp$  and  $u \in \partial\sigma_A(x^*)$ . Because  $\sigma_A$  is differentiable at  $x^*$ , by Proposition 9 we obtain that  $\nabla\sigma_A(x^*) = u \in A$ .  $\square$

Note that we cannot obtain the convexity of  $A$  in Corollary 11 under its hypothesis.

**Example 2** Consider  $X := \mathbb{R}^2$  endowed with the Euclidean norm and the sets  $A_1 := \{(x, y) \in X \mid \|(x, y)\| = 1\}$ ,  $A_2 := \{x \in X \mid 1/2 \leq \|(x, y)\| \leq 1\}$ ,  $A_3 := \{(x, y) \in X \mid x > 0, y = 1/x\}$  and  $A_4 := \{(x, y) \in X \mid x > 0, 1/x \leq y \leq 2/x\}$ . Then  $\sigma_{A_1}(u, v) = \sigma_{A_2}(u, v) = \|(u, v)\| = \sqrt{u^2 + v^2}$  and  $\sigma_{A_3}(u, v) = \sigma_{A_4}(u, v) = -2\sqrt{uv}$  for  $u, v \leq 0$ ,  $\sigma_{A_3}(u, v) = \sigma_{A_4}(u, v) = +\infty$  otherwise. Clearly,  $\sigma_{A_i}$  is differentiable on  $\text{dom } \partial\sigma_{A_i} \setminus (\text{lin}_0 A_i)^\perp = \text{int}(\text{dom } \sigma_{A_i}) \setminus \{0\}$ . Note that  $A_i = \text{cl}(\text{int } A_i)$  for  $i \in \{2, 4\}$ .

These simple examples show that there is no hope to get the convexity of  $A$  from the differentiability of  $\sigma_A$  on  $\text{dom } \partial\sigma_A \setminus (\text{lin}_0 A)^\perp$  or on  $\text{int}(\text{dom } \sigma_A) \setminus \{0\}$  in the case in which  $A$  is bounded. Even for  $A$  unbounded one needs supplementary conditions. In the sequel we concentrate on the case in which  $A$  is unbounded.

In Theorem 15 below we provide a supplementary condition on  $A$  to be added in Corollary 11 in order to get the convexity of  $A$ . First we establish an auxiliary result.

**Lemma 12** *Assume that  $A_\infty$  is a pointed convex cone and  $A = A + A_\infty$ . Then  $\text{conv } A$  is closed.*

Proof. Let  $u \in \overline{\text{conv}} A$ . Using the Carathéodory theorem, we find  $(\lambda_n^k)_{n \geq 1} \subset [0, 1]$  and  $(u_n^k)_{n \geq 1} \subset A$  for  $k \in \overline{1, p+1}$  such that  $\sum_{k=1}^{p+1} \lambda_n^k = 1$ ,  $u_n := \sum_{k=1}^{p+1} \lambda_n^k u_n^k \rightarrow u$ . As in the proof of Proposition 8, we have that the sequences  $(\lambda_n^k u_n^k)_{n \geq 1}$  are bounded. In the contrary case we may assume that

$$\|\lambda_n^1 u_n^1\| \leq \|\lambda_n^2 u_n^2\| \leq \dots \leq \|\lambda_n^{p+1} u_n^{p+1}\| \quad \forall n \geq 1$$

and  $\|\lambda_n^{p+1} u_n^{p+1}\| \rightarrow \infty$  as  $n \rightarrow \infty$ . Taking subsequences if necessary, we may assume that  $\|\lambda_n^{p+1} u_n^{p+1}\|^{-1} \lambda_n^k u_n^k \rightarrow v_k$  for  $k \in \overline{1, p+1}$ . Since  $\lambda_n^k \in [0, 1]$  and  $\|\lambda_n^{p+1} u_n^{p+1}\| \rightarrow \infty$  we have that  $\|\lambda_n^{p+1} u_n^{p+1}\|^{-1} \lambda_n^k \rightarrow 0$ . Since  $u_n^k \in A$ , we obtain that  $v_k \in A_\infty =: K$  for every  $k \in \overline{1, p+1}$ . From  $\sum_{k=1}^{p+1} \lambda_n^k u_n^k \rightarrow u$  we get  $v_1 + \dots + v_{p+1} = 0$ . Since  $K$  is convex we get  $v_1 + \dots + v_p = -v_{p+1} \in K \cap (-K)$ . Because  $K$  is pointed we get the contradiction  $v_{p+1} = 0$ . Hence the sequences  $(\lambda_n^k u_n^k)_{n \geq 1}$  are bounded. We may assume that  $\lambda_n^k \rightarrow \lambda^k$  and  $\lambda_n^k u_n^k \rightarrow v^k \in X$  for every  $k \in \overline{1, p+1}$ ; moreover, we may assume that  $\lambda^1 \leq \lambda^2 \leq \dots \leq \lambda^{p+1}$ . Since  $\sum_{k=1}^{p+1} \lambda^k = 1$ ,  $\lambda^{p+1} > 0$ . If  $\lambda^k > 0$  then  $u_n^k \rightarrow u^k := (\lambda^k)^{-1} v^k \in A$ . If  $\lambda^1 > 0$  then  $u = \sum_{k=1}^{p+1} \lambda^k u^k \in \text{conv } A$ . If  $\lambda^1 = 0$  take  $k_0 \in \overline{1, p}$  such that  $\lambda^{k_0} = 0$  and  $\lambda^{k_0+1} > 0$ . Then  $u = \bar{v} + \bar{u}$  with  $\bar{v} := \sum_{k=1}^{k_0} \lambda^k v^k \in K$ ,  $\bar{u} := \sum_{k=k_0+1}^{p+1} \lambda^k u^k$ , and so  $u = \sum_{k=k_0+1}^{p+1} \lambda^k (u^k + \bar{v}) \in \text{conv } A$ . It follows that  $\overline{\text{conv}} A \subset \text{conv } A$ .  $\square$

A nice application of the preceding lemma is the fact that the convex hull of the epigraph of a proper lsc 1-coercive function  $f : \mathbb{R}^p \rightarrow \overline{\mathbb{R}}$  is closed, result which can be found in [14], [7], [2]. First we give the next result which is probably known.



**Proposition 13** *Let  $f : X \rightarrow \overline{\mathbb{R}}$  be a proper lsc function. Then  $f$  is 1-coercive (that is  $\lim_{\|x\| \rightarrow \infty} f(x)/\|x\| = \infty$ ) if and only if  $f_\infty = \iota_{\{0\}}$ .*

*Proof.* Assume first that  $\lim_{\|x\| \rightarrow \infty} f(x)/\|x\| = \infty$  and take  $(u, \alpha) \in (\text{epi } f)_\infty$ . Then there exist the sequences  $(t_n)_{n \geq 1} \subset \mathbb{P}$  and  $((x_n, \lambda_n))_{n \geq 1} \subset \text{epi } f$  such that  $t_n \rightarrow 0$  and  $t_n(x_n, \lambda_n) \rightarrow (u, \alpha)$ . Suppose that  $u \neq 0$ . Then  $\|x_n\| \rightarrow \infty$ , and so  $f(x_n)/\|x_n\| \rightarrow \infty$ . Since  $f(x_n)/\|x_n\| \leq (t_n \lambda_n)/\|t_n x_n\| \rightarrow \alpha/\|u\|$ , we get the contradiction  $\infty \leq \alpha/\|u\|$ . Hence  $u = 0$ . If  $\alpha < 0$  then  $t_n \lambda_n \leq \alpha/2 < 0$  for large  $n$ , and so  $f(x_n) \leq \alpha/(2t_n)$  for such  $n$ . Hence  $f(x_n) \rightarrow -\infty$ . Because  $f$  is 1-coercive, we obtain that  $(x_n)$  is bounded, and so, passing to a subsequence if necessary, we assume that  $x_n \rightarrow x$ . Since  $f$  is lsc, we get the contradiction  $-\infty < f(x) \leq \liminf f(x_n) = -\infty$ .

Assume now that  $\liminf_{\|x\| \rightarrow \infty} f(x)/\|x\| < \infty$ . Then there exist a sequence  $(x_n)_{n \geq 1} \subset \mathbb{R}_+$  and  $\alpha \in \mathbb{R}$  such that  $\|x_n\| \rightarrow \infty$  and  $f(x_n)/\|x_n\| \leq \alpha$  for every  $n$ . Passing to a subsequence if necessary, we assume that  $x_n/\|x_n\| \rightarrow u$  ( $\neq 0$ ). Since  $(x_n, \alpha\|x_n\|) \in \text{epi } f$  and  $0 < t_n := \|x_n\|^{-1} \rightarrow 0$ , we get  $(u, \alpha) = \lim t_n(x_n, \alpha\|x_n\|) \in (\text{epi } f)_\infty$ . Hence  $(\text{epi } f)_\infty \neq \{0\} \times \mathbb{R}_+$ . The proof is complete.  $\square$

**Corollary 14** *Let  $f : X \rightarrow \overline{\mathbb{R}}$  be a proper lsc function. Assume that  $f_\infty$  is a proper convex (hence sublinear) function such that  $f_\infty(u) + f_\infty(-u) > 0$  for all  $u \in X \setminus \{0\}$  and  $f(x+u) \leq f(x) + f_\infty(u)$  for all  $x, u \in X$ ; then  $\text{conv}(\text{epi } f)$  is closed. In particular,  $\text{conv}(\text{epi } f)$  is closed if  $f$  is 1-coercive.*

*Proof.* Since  $f_\infty$  is convex we have that  $(\text{epi } f)_\infty$  is a closed convex cone. Take  $(u, \alpha) \in (\text{epi } f)_\infty \cap -(\text{epi } f)_\infty$ . Then  $0 = f_\infty(0) \leq f_\infty(u) + f_\infty(-u) \leq \alpha + (-\alpha) = 0$ , and so  $u = 0$ . Then  $0 = f_\infty(0) \leq \min\{\alpha, -\alpha\}$ , whence  $\alpha = 0$ . Hence  $(\text{epi } f)_\infty$  is pointed. Using Lemma 8 we obtain that  $\text{conv}(\text{epi } f)$  is closed.  $\square$

Another example (besides the epigraph of a proper lsc 1-coercive function) of set  $A$  verifying the hypothesis of Lemma 12 is when  $A = A + K \subset K$  with  $K \subset X$  a proper closed convex pointed cone because in this case  $A_\infty = (\overline{\text{conv}} A)_\infty = K$ .

In the rest of this section we assume that the subsets  $A$  and  $K$  of  $X$  verify the following condition

(H)  *$K$  is a pointed closed convex cone with nonempty interior and  $A$  is a closed nonempty set such that  $A = A + K \subset K \setminus \{0\}$ .*

In this case  $\text{dom } \sigma_A = K^-$  and  $\text{int}(\text{dom } \sigma_A) = \text{int } K^- = -K^\#$ . Moreover,

$$\emptyset \neq \text{int } A = A + \text{int } K = \text{int } K \cap \text{int}_K A \quad \text{and} \quad A = \text{cl}(A \cap \text{int } K); \quad (10)$$

it follows that  $\text{int}_K A = \text{int } A$ , and so  $\text{bd}_K A = \text{bd } A$ , if  $A \subset \text{int } K$ .

**Theorem 15** *Assume that*

$$A + (K \setminus \{0\}) \subset \text{int}_K A. \quad (11)$$

*If  $\sigma_A$  is differentiable on  $\text{int } K^-$  then  $A$  is convex and*

$$\forall a, a' \in A \cap \text{int } K, \quad a \neq a', \quad \forall \lambda \in (0, 1) : \lambda a + (1 - \lambda)a' \in \text{int } A. \quad (12)$$

*Conversely, if  $\dim X = 2$  and (12) holds, then  $A$  is convex and (7) is verified; therefore,  $\sigma_A$  is differentiable on  $\text{int } K^-$ .*



Proof. Because  $A + (K \setminus \{0\}) \subset \text{int}_K A \subset A$ , we have that  $A + K = A$ . Then  $\text{conv } A = (\text{conv } A) + K$ , and so  $\text{int}(\text{conv } A) = (\text{conv } A) + \text{int } K$ .

Assume now that  $\sigma_A$  is differentiable on  $\text{int } K^-$ . As seen above,  $A_\infty = K$  and  $A = A + K$ ; hence  $\text{conv } A$  is closed by Lemma 12.

Let us prove that  $\text{int } K \cap \text{conv } A \subset A$ . Take first  $\bar{u} \in \text{int } K \cap \text{bd}(\text{conv } A)$  and consider  $k \in K \setminus \{0\}$ . Since  $\bar{u} \in \text{conv } A$ ,  $\bar{u} = \sum_{i \in I} \lambda_i u_i$  for some nonempty finite set  $I$ ,  $(\lambda_i)_{i \in I} \subset (0, 1)$  with  $\sum_{i \in I} \lambda_i = 1$ , and  $(u_i)_{i \in I} \subset A$ . Since  $u_i + k \in \text{int}_K A$  by our hypothesis, there exists  $\mu_i \in (0, 1)$  such that  $\mu(u_i + k) \in A$  for every  $\mu \in [\mu_i, 1]$ . Taking  $\bar{\mu} := \max\{\mu_i \mid i \in I\} \in (0, 1)$ , we have that  $\bar{\mu}(u_i + k) \in A$  for every  $i \in I$ . It follows that  $\bar{\mu}(\bar{u} + k) = \sum_{i \in I} \lambda_i \bar{\mu}(u_i + k) \in \text{conv } A$ , and so

$$\bar{u} + k = \bar{\mu}(\bar{u} + k) + (1 - \bar{\mu})(\bar{u} + k) \in \text{conv } A + \text{int } K \subset \text{int}(\text{conv } A).$$

Hence  $\bar{u} + (K \setminus \{0\}) \subset \text{int}(\text{conv } A)$ . Since  $\bar{u} \in \text{bd}(\text{conv } A)$ , there exists  $x^* \in X^* \setminus \{0\}$  such that  $\sigma_A(x^*) = \langle \bar{u}, x^* \rangle$ . Because  $x^* \neq 0$ , we have that  $\langle \bar{u}, x^* \rangle > \langle u, x^* \rangle$  for every  $u \in \text{int}(\text{conv } A)$ . Because  $\bar{u} + (K \setminus \{0\}) \subset \text{int}(\text{conv } A)$  we get  $\langle k, x^* \rangle < 0$  for every  $k \in K \setminus \{0\}$ , and so  $x^* \in -K^\# = \text{int}(\text{dom } \sigma_A)$ . Because  $\sigma_A$  is differentiable at  $x^*$ , by Proposition 9 we obtain that  $\nabla \sigma_A(x^*) = \bar{u} \in A$ .

Take now  $\bar{u} \in \text{int } K \cap \text{conv } A$  and consider  $\alpha := \min\{\gamma > 0 \mid \gamma \bar{u} \in \text{conv } A\} \in (0, 1]$ ; clearly  $\alpha \bar{u} \in \text{int } K \cap \text{bd}(\text{conv } A)$ . By the argument above we have that  $\alpha \bar{u} \in A$ . If  $\alpha = 1$  then  $\bar{u} = \alpha \bar{u} \in A$ . If  $\alpha \in (0, 1)$  then  $\bar{u} = \alpha \bar{u} + (1 - \alpha)\bar{u} \in A + \text{int } K \subset A$ . Therefore,  $\text{int } K \cap \text{conv } A \subset A$ , whence  $\text{conv } A = K \cap \text{conv } A = \text{cl}(\text{int } K \cap \text{conv } A) \subset \text{cl } A = A$ . Therefore,  $A$  is convex.

Assume that (12) does not hold. Then there exist  $a, a' \in A \cap \text{int } K$  with  $a \neq a'$  and  $\lambda \in (0, 1)$  such that  $\bar{a} := \lambda a + (1 - \lambda)a' \in \text{bd } A$ ; of course,  $\bar{a} \in \text{int } K$ . By (11) we have that  $\bar{a} + (K \setminus \{0\}) \subset \text{int } K \cap \text{int}_K A = \text{int } A$ . Because  $A$  is convex, as above (with  $\bar{a}$  instead of  $\bar{u}$ ), there exists  $x^* \in -K^\# = \text{int } K^-$  such that  $\nabla \sigma_A(x^*) = \bar{a}$ . We have that

$$\sigma_A(x^*) = \langle \bar{a}, x^* \rangle = \lambda \langle a, x^* \rangle + (1 - \lambda) \langle a', x^* \rangle \leq \lambda \sigma_A(x^*) + (1 - \lambda) \sigma_A(x^*) = \sigma_A(x^*),$$

whence  $\langle a, x^* \rangle = \langle a', x^* \rangle = \sigma_A(x^*)$ . Hence  $a, a' \in \partial \sigma_A(x^*) = \{\bar{a}\}$  which yields the contradiction  $a = a'$ . Therefore, (12) holds.

Assume now that  $\dim X = 2$  and (12) holds. Hence  $K = \mathbb{R}_+ x_1 + \mathbb{R}_+ x_2$  with  $x_1, x_2 \in X$  linearly independent. From (12) we have obviously that  $A \cap \text{int } K$  is convex, and so, from the last equality in (10) we get the convexity of  $A$ .

Let  $x, x' \in A$  with  $x \neq x'$  and  $\lambda \in (0, 1)$ . Set  $x'' := \lambda x + (1 - \lambda)x'$ . Assume that  $x' \in \text{int } K$ ; then  $u := \frac{1}{2}x + \frac{1}{2}x'' = \frac{1+\lambda}{2}x + \frac{1-\lambda}{2}x' \in A \cap \text{int } K$ , and so  $x'' = \frac{2\lambda}{1+\lambda}u + \frac{1-\lambda}{1+\lambda}x' \in \text{int } A = A + \text{int } K \subset A + (K \setminus \{0\}) = A + (A_\infty \setminus \{0\})$ .

Assume now that  $x, x' \in \text{bd } K = \mathbb{R}_+ x_1 \cup \mathbb{R}_+ x_2$ . If  $x, x' \in \mathbb{R}_+ x_1$  then  $x = \alpha x_1$ ,  $x' = \alpha' x_1$  with  $\alpha, \alpha' > 0$ . Letting  $\alpha > \alpha'$  we have that  $x'' = x' + \lambda(\alpha - \alpha')x_1 \in A + (K \setminus \{0\})$ . If  $x \in \mathbb{R}_+ x_1$  and  $x' \in \mathbb{R}_+ x_2$  then  $x = \alpha x_1$ ,  $x' = \alpha' x_2$  with  $\alpha, \alpha' > 0$ . Take  $u := \frac{\lambda}{2}x + \frac{1-\lambda}{2}x' \in A \cap (\mathbb{P}x_1 + \mathbb{P}x_2)$  and  $v := \frac{\lambda}{2}x + \frac{2-\lambda}{2}x' \in A \cap (\mathbb{P}x_1 + \mathbb{P}x_2)$ . Since  $\text{int } K = \mathbb{P}x_1 + \mathbb{P}x_2$ , we obtain that  $x'' = \lambda u + (1 - \lambda)v \in \text{int } A \subset A + (A_\infty \setminus \{0\})$ . Hence (7) is verified. Applying Proposition 4 we obtain that  $\sigma_A$  is differentiable on  $\text{int } K^-$ . The proof is complete.  $\square$

Note that  $A := x_0 + K$  with  $x_0 \in K \setminus \{0\}$  verifies condition (H) and  $\sigma_A$  is differentiable on  $\text{int } K^-$ . However, (11) is not verified. So, condition (11) is far from being necessary for the differentiability of  $\sigma_A$ .

(Assume that  $x_0 \in K \setminus \{0\}$  and  $A := x_0 + K$  verifies (11). If  $x_0 \in \text{int } K$ , then  $A \subset \text{int } K$ , and so, by (10),  $\text{int } A = \text{int}_K A$ ; thus  $x_0 + (K \setminus \{0\}) \subset \text{int } A = x_0 + \text{int } K$ , whence

$K \setminus \{0\} \subset \text{int } K$ , a contradiction. Hence  $x_0 \in \text{bd } K$ . Then there exists  $u \in \text{int } K$  and  $t > 0$  such that  $(1+t)u - tx_0 \notin K$ . Otherwise  $\text{int } K - x_0 \subset K$ , whence  $K \subset x_0 + K$ . This implies the contradiction  $-x_0 \in K$ . It follows that there exist  $u \in \text{int } K$  and  $t > 0$  such that  $u_0 := (1+t)u - tx_0 \in \text{bd } K$ . Clearly,  $u_0 \neq 0$ ; else  $x_0 = (1+t^{-1})u \in \text{int } K$ , a contradiction. It follows that  $x_0 + t^{-1}u_0 = (1+t^{-1})u \in \text{int } K \cap \text{int}_K A = \text{int } A = x_0 + \text{int } K$ , and so  $u_0 \in \text{int } K$ , a contradiction.)

A condition which is slightly stronger than (12) is sufficient for the differentiability of  $\sigma_A$ .

**Proposition 16** *Assume that*

$$\forall a, a' \in A, a \neq a', \forall \lambda \in (0, 1) : \lambda a + (1 - \lambda)a' \in \text{int}_K A. \quad (13)$$

*Then  $A$  is convex,  $\sigma_A$  is differentiable on  $\text{int } K^-$  and (11) holds.*

Proof. From (13) and  $\text{int}_K A \subset A$  we get the convexity of  $A$ . Let now  $a \in A$  and  $k \in K \setminus \{0\}$ . Then  $a \neq a + 2k \in A$ . From (13) we obtain that  $a + k = \frac{1}{2}a + \frac{1}{2}(a + 2k) \in \text{int}_K A$ , and so (11) holds.

Assume that  $\sigma_A$  is not differentiable on  $\text{int } K^-$ . Then there exist  $\bar{x}^* \in \text{int } K^-$  and  $a, a' \in \partial\sigma_A(\bar{x}^*) \subset \text{bd } A \subset A$  such that  $a \neq a'$ . By (13) we have that  $0 \neq \bar{a} := \frac{1}{2}a + \frac{1}{2}a' \in \partial\sigma_A(\bar{x}^*) \cap \text{int}_K A$ . In particular,  $\langle \bar{a}, \bar{x}^* \rangle = \sigma_A(\bar{x}^*)$ . Then there exists  $\alpha \in (0, 1)$  such that  $\alpha\bar{a} \in A \subset K \setminus \{0\}$ , and so  $\langle \bar{a}, \bar{x}^* \rangle < 0$ . This is in contradiction with  $\langle \bar{a}, \bar{x}^* \rangle = \sigma_A(\bar{x}^*) \geq \langle \alpha\bar{a}, \bar{x}^* \rangle$ . Therefore,  $\sigma_A$  is differentiable on  $\text{int } K^-$ .  $\square$

The example above (that is  $A := x_0 + K$  with  $x_0 \in K \setminus \{0\}$ ) shows that the condition (13) is not necessary for the differentiability of  $\sigma_A$  on  $\text{int } K^-$ .

Applying Theorem 5, we obtain that (13)  $\Rightarrow$  (8); the advantage of (13) is that this condition is more intuitive (and quite easy to be verified).

As seen above,  $\text{int } A = \text{int}_K A$  and  $\text{bd } A = \text{bd}_K A$  when  $A \subset \text{int } K$ ; in this case condition (11) becomes

$$A + (K \setminus \{0\}) \subset \text{int } A. \quad (14)$$

Conversely, if  $A$  verifies (14), then  $A \subset \text{int } K$ . Indeed, in the contrary case there exists  $k \in A \setminus \text{int } K \subset K \setminus \{0\}$ . From (14) we get  $2k = k + k \in \text{int } A \subset \text{int } K$ , whence the contradiction  $k \in \text{int } K$ .

The next result characterizes the differentiability of  $\sigma_A$  for  $A \subset \text{int } K$ .

**Corollary 17** *Assume that  $A \subset \text{int } K$  and (14) holds. Then  $\sigma_A$  is differentiable on  $\text{int } K^-$  if and only if  $A$  is convex and*

$$\forall a, a' \in \text{bd } A, a \neq a', \forall \lambda \in (0, 1) : \lambda a + (1 - \lambda)a' \notin \text{bd } A. \quad (15)$$

Proof. From the observation above we have that  $\text{int } A = \text{int}_K A$ , and so (11) holds. Assume first that  $\sigma_A$  is differentiable on  $\text{int } K^-$ . By Theorem 15 we have that  $A$  is convex and (12) holds. Take  $a, a' \in \text{bd } A \subset \text{int } K$  with  $a \neq a'$  and  $\lambda \in (0, 1)$ . By (12) we have that  $\bar{a} := \lambda a + (1 - \lambda)a' \in \text{int } A$ . Therefore,  $\bar{a} \notin \text{bd } A$ , and so (15) holds.

Assume now that  $A$  is convex and (15) holds. Then  $A$  is strictly convex, and so, using Theorem 1,  $\sigma_A$  is differentiable on  $\text{dom } \partial\sigma_A \setminus (\text{lin}_0 A)^\perp = \text{dom } \partial\sigma_A \setminus \{0\} \supset \text{int } K^-$ .  $\square$

Note that for obtaining Proposition 16, or relation (12) when  $A$  is convex and  $\sigma_A$  is differentiable on  $\text{int } K^- (= \text{int}(\text{dom } \sigma_A))$  in Theorem 15, it is not possible to use anyone of the results in Section 3.

In the sequel we use the convention  $0A := A_\infty = K$ . Then

$$[\alpha, \infty)A = \alpha A \quad \forall \alpha \in \mathbb{R}_+. \quad (16)$$

Indeed, the equality is obvious for  $\alpha = 0$  (since  $A \subset K$  and  $0A = K$ ). Let  $\alpha > 0$  and take  $t \in [\alpha, \infty)$ ,  $x \in A$ ; then  $tx = \alpha[x + (t/\alpha - 1)x] \in \alpha(A + K) = \alpha A$ .

The set  $\{t \geq 0 \mid x \in tA\}$  is a compact interval containing 0. Using the facts that  $0A = K$ ,  $0 \notin A = \text{cl } A$ , and (16), we obtain that the function

$$F_A : K \rightarrow \mathbb{R}_+, \quad F_A(x) := \max \{t \geq 0 \mid x \in tA\}. \quad (17)$$

is well defined. The function  $F_A$  is the restriction to  $K$  of the function  $\beta_A$  considered in [9] in a more general setting.

In the following proposition we mention some properties of  $F_A$ .

**Theorem 18** *Let  $A$  and  $K$  be as above.*

(i)  $F_A(tx) = tF_A(x)$  for all  $x \in K$  and  $t \geq 0$ ,  $x \in F_A(x) \cdot A$  for every  $x \in K$ , and

$$\text{int } K \subset \mathbb{P}A = \{x \in K \mid F_A(x) > 0\}. \quad (18)$$

(ii) One has

$$\{x \in K \mid F_A(x) \geq \gamma\} = \gamma A \quad \forall \gamma \in \mathbb{R}_+. \quad (19)$$

Consequently,  $F_A$  is upper semicontinuous (usc for short).

(iii)  $A$  is convex iff  $F_A$  is quasiconcave iff  $F_A$  is concave.

(iv)  $F_A(x') \geq F_A(x)$  for all  $x, x' \in K$  with  $x' \geq_K x$ .

(v)  $F_A$  is continuous on  $(K \setminus \mathbb{P}A) \cup \text{int } K$ . Therefore,  $F_A$  is continuous whenever  $A \subset \text{int } K$ .

(vi) If  $K$  is polyhedral, then  $F_A$  is continuous.

(vii) a) (14) holds iff  $x' \geq_K x \in \text{int } K$  implies  $F_A(x') > F_A(x)$ . b) (11) holds iff  $F_A$  is continuous and  $x' \geq_K x \in \mathbb{P}A$  implies  $F_A(x') > F_A(x)$ .

(viii) a) (12) holds iff  $F_A$  is strictly quasi-concave on  $\text{int } K$ . b) If (13) holds, then  $F_A$  is strictly quasi-concave on  $\mathbb{P}A$ ; conversely, if  $F_A$  is continuous and strictly quasi-concave on  $\mathbb{P}A$  then (13) holds.

Proof. (i) Since  $0 \in 0A \setminus \mathbb{P}A$  we have that  $F_A(0) = 0$ . The relation  $F_A(tx) = tF_A(x)$  for  $x \in K$  and  $t > 0$  follows immediately from the definition. Also the relation  $x \in F_A(x) \cdot A$  follows from the very definition of  $F_A$  (the supremum being attained).

The equality in (18) is obvious. Assume that there exists  $x \in (\text{int } K) \setminus \mathbb{P}A$ . Then  $\mathbb{P}x \cap A = \emptyset$ , and so  $\mathbb{P}x \cap (a + K) = \emptyset$ , where  $a \in A$  is a fixed element. Since  $\mathbb{P}x$  and  $a + K$  are convex sets, there exists  $x^* \in X^* \setminus \{0\}$  such that  $\langle tx, x^* \rangle \leq \langle a + u, x^* \rangle$  for all  $t > 0$  and  $u \in K$ . It follows that  $x^* \in K^+$  and  $\langle x, x^* \rangle \leq 0$ . Since  $x \in \text{int } K$  we get the contradiction  $x^* = 0$ .

(ii) The inclusion  $\supset$  in (19) is obvious. The converse inclusion is immediate from (16) and the fact that  $x \in F_A(x) \cdot A$  for every  $x \in K$ . Because  $\gamma A$  is closed for every  $\gamma \geq 0$ , from (19) we obtain that  $F_A$  is usc.

(iii) The first equivalence follows from (19). Moreover, if  $F_A$  is concave, clearly  $F_A$  is quasiconcave. Assume that  $F_A$  is quasiconcave. Consider  $f : X \rightarrow \overline{\mathbb{R}}$  defined by  $f(x) := -F_A(x)$  for  $x \in K$  and  $f(x) := +\infty$  for  $x \in X \setminus K$ . Then  $f$  is quasiconvex. Because  $F_A$  is usc and  $\text{dom } f = K$  is closed we have that  $f$  is lsc. Moreover, from (i) we have that  $f(tx) = tf(x)$  for all  $t \in \mathbb{P}$  and  $x \in X$ , and  $\{x \in X \mid f(x) < 0\} = \{x \in K \mid F_A(x) > 0\} \supset \text{int } K$ , whence

$\text{dom } f = K = \text{cl}\{x \in X \mid f(x) < 0\}$ . Applying [15, Thm. 2.2.2] we obtain that  $f$  is sublinear; in particular,  $f$  is convex, and so  $F_A$  is concave.

(iv) Take  $x, x' \in K$  with  $x' \geq_K x$ . If  $\gamma := F_A(x) = 0$  then clearly  $F_A(x') \geq \gamma$ . Else,  $\gamma > 0$  and  $x \in \gamma A$ ; hence  $x' \in x + K \subset \gamma A + K = \gamma(A + K) = \gamma A$ , and so  $F_A(x') \geq \gamma$  by (ii).

(v) Let  $x \in \text{int } K$ ; then  $\gamma := F_A(x) > 0$ . Take  $0 < \mu < \gamma$ . Then

$$\mu^{-1}x = \gamma^{-1}x + (\mu^{-1} - \gamma^{-1})x \in A + \text{int } K \subset \text{int } A.$$

It follows that  $A$  is a neighborhood of  $\mu^{-1}x$ , whence  $V := \mu A$  is a neighborhood of  $x$ . Since  $F_A(x') \geq \mu$  for every  $x' \in V$ , we have that  $F_A$  is lsc at  $x$ . By (ii) we get the continuity of  $F_A$  at  $x$ .

Take  $x \in K \setminus \mathbb{P}A$ ; from (18) we have that  $F_A(x) = 0 = \inf F_A$ , and so  $F_A$  is lsc at  $x$ . Since  $F_A$  is usc, we have that  $F_A$  is continuous at  $x$ . Hence  $F_A$  is continuous at any  $x \in (K \setminus \mathbb{P}A) \cup \text{int } K$ .

If  $A \subset \text{int } K$ , then  $\mathbb{P}A = \text{int } K$ , and so  $(K \setminus \mathbb{P}A) \cup \text{int } K = K$ .

(vi) There exists  $(x_i^*)_{i \in \overline{1, m}} \subset X^* \setminus \{0\}$  such that  $K = \{x \in X \mid \langle x, x_i^* \rangle \geq 0 \ \forall i \in \overline{1, m}\}$ ,  $\text{int } K = \{x \in X \mid \langle x, x_i^* \rangle > 0 \ \forall i \in \overline{1, m}\} \neq \emptyset$  and  $\cap_{i=1}^m \ker x_i^* = \{0\}$ .

Take  $a \in \mathbb{P}A$  and set  $I := \{i \in \overline{1, m} \mid \langle a, x_i^* \rangle > 0\} (\neq \emptyset)$ . Let  $\gamma := F_A(a) (> 0)$  and take  $\mu \in (0, \gamma)$ ; clearly  $a \in \gamma A$ . There exists a neighborhood  $V$  of  $a$  such that  $\langle x, x_i^* \rangle \geq \mu \gamma^{-1} \langle a, x_i^* \rangle$  for all  $x \in V$  and  $i \in I$ . Then for each  $x \in K \cap V$  and each  $i \in \overline{1, m}$  we have that  $\langle \mu^{-1} \gamma x - a, x_i^* \rangle \geq 0$ . Thus  $\mu^{-1}x \in \gamma^{-1}a + K \subset A$ . Hence  $F_A(x) \geq \mu$  for every  $x \in K \cap V$ , and so  $F_A$  is lsc at  $x$ . It follows that  $F_A$  is continuous at  $x$ .

(vii) b) “ $\implies$ ” If  $x \in K \setminus \mathbb{P}A$  then  $F_A(x) = 0 = \inf F_A$ , and so  $F_A$  is lsc (hence continuous by (ii)) at  $x$ . Let  $x \in \mathbb{P}A (\subset K \setminus \{0\})$  and take  $\gamma := F_A(x) > 0$ . Consider  $0 < \mu < \gamma$ . Then, as in (v), we get  $\mu^{-1}x \in A + K \setminus \{0\} \subset \text{int}_K A$ . Hence  $V := \mu A$  is a neighborhood (in  $K$ ) of  $x$ . Since  $F_A(x') \geq \mu$  for every  $x' \in V$ , we obtain that  $F_A$  is lsc at  $x$ .

Let now  $x' \geq_K x \in \mathbb{P}A$  and take  $\gamma := F_A(x) > 0$ ; then  $x' = x + k (\in K \setminus \{0\})$  for some  $k \in K \setminus \{0\}$ . It follows that  $\gamma^{-1}x' = \gamma^{-1}x + \gamma^{-1}k \in \text{int}_K A$ , and so there exists a neighborhood  $V$  of  $\gamma^{-1}x'$  such that  $K \cap V \subset A$ . Then there exists  $\mu > \gamma$  such that  $\mu^{-1}x' \in K \cap V \subset A$ , whence  $F_A(x') \geq \mu > \gamma$ .

“ $\impliedby$ ” Consider  $x \in A$ ,  $k \in K \setminus \{0\}$  and  $x' := x + k$ . Then  $x' \geq_K x \in \mathbb{P}A$ , and so  $F_A(x') > F_A(x) \geq 1$ . Because  $F_A$  is continuous at  $x'$ , there exists a neighborhood  $V$  of  $x'$  such that  $F_A(x'') \geq 1$  for every  $x'' \in K \cap V$ . It follows that  $K \cap V \subset A$ , and so  $x' \in \text{int}_K A$ .

The proof of a) is similar.

(viii) b) Assume that (13) holds. Take  $x, x' \in \mathbb{P}A$  with  $x \neq x'$  and  $\lambda \in (0, 1)$ . We may (and do) assume that  $F_A(x') \geq F_A(x) =: \gamma > 0$ . Then  $x', x \in \gamma A$ . It follows that  $\gamma^{-1}x'' \in \text{int}_K A$ , where  $x'' := (\lambda x + (1 - \lambda)x')$ . As in the proof of (vii) b) above, there exists  $\mu > \gamma$  such that  $\mu^{-1}x'' \in A$ , whence  $F_A(x'') \geq \mu > \gamma = F_A(x)$ . Hence  $F_A$  is strictly quasi-concave on  $\mathbb{P}A$ .

Assume now that  $F_A$  is continuous and strictly quasi-concave on  $\mathbb{P}A$ . Take  $x, x' \in A$  with  $x \neq x'$  and  $\lambda \in (0, 1)$ . Then  $F_A(x), F_A(x') \geq 1$ . Because  $F_A$  is strictly quasi-concave, we have that  $F_A(x'') > 1$ , where  $x'' := \lambda x + (1 - \lambda)x'$ . From (18) we have that  $x'' \in \mathbb{P}A$ , and so  $F_A$  is continuous at  $x''$ . Then there exists a neighborhood  $V$  of  $x''$  such that  $F_A(y) \geq 1$  for every  $y \in K \cap V$ , whence  $K \cap V \subset A$ . This shows that  $x'' \in \text{int}_K A$ .

The proof of a) is similar; take into account that  $F_A$  is continuous on  $\text{int } K$ .  $\square$

The next examples show that in several results the converse implications are not valid.

**Example 3** (a) The set  $A := a + K$  with  $a \in K \setminus \{0\}$  does not verify condition (11), but  $\sigma_A$  is differentiable on  $\text{int } K^-$ . Also the condition (14) is not necessary for the differentiability of  $\sigma_A$  when  $A \subset \text{int } K$  (take  $a \in \text{int } K$ ).

(b) The set  $A := \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 + x_2 \geq 1\}$  verifies condition (11) for  $K := \mathbb{R}_+^2$ , but  $\sigma_A$  is not differentiable on  $\text{int } K^- = -\mathbb{R}_{++}^2$ .

(c) Let  $A := a + K$  with  $K := \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \geq \sqrt{(x_1)^2 + (x_2)^2}\}$  and  $a \in (\text{bd } K) \setminus \{0\}$ . Then  $\mathbb{P}A = \mathbb{P}a \cup \text{int } K$  and  $F_A$  is not continuous at  $x \in K$  iff  $x \in \mathbb{P}a$ . (This fact shows that the polyhedrality of  $K$  in Theorem 18 (vi) is essential.) Moreover,  $F_A$  is sublinear.

(d) Let  $A := a + K$  with  $K := \{x \in X \mid \langle x, x_i^* \rangle \geq 0 \ \forall i \in \overline{1, m}\}$ , where  $(x_i^*)_{i \in \overline{1, m}} \subset X^* \setminus \{0\}$  are such that  $\cap_{i=1}^m \ker x_i^* = \{0\}$  and  $(\text{int } K =) \{x \in X \mid \langle x, x_i^* \rangle > 0 \ \forall i \in \overline{1, m}\} \neq \emptyset$ , and  $a \in K \setminus \{0\}$ . Then  $F_A(x) = \min \{\langle x, x_i^* \rangle / \langle a, x_i^* \rangle \mid \langle a, x_i^* \rangle > 0\}$  for  $x \in K$ . For  $K := \mathbb{R}_{++}^p$ ,  $F_A$  is the Leontieff production function (and  $F_A(x) = \min \{x_i/a_i \mid a_i > 0\}$ ).

(a) This example was considered before Proposition 16. Moreover,  $\sigma_A(x^*) = \langle a, x^* \rangle + \iota_{K^-}(x^*)$ .

(b) In this case  $A + (K \setminus \{0\}) = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 + x_2 > 1\} = \text{int}_K A$  and  $\sigma_A(u_1, u_2) = \max\{u_1, u_2\} + \iota_{-\mathbb{R}_+^2}(u_1, u_2)$ .

(c) Let  $a := (a_1, a_2, a_3) \in (\text{bd } K) \setminus \{0\}$ ; then  $a_3 = \sqrt{(a_1)^2 + (a_2)^2} > 0$ . Then for every  $x := (x_1, x_2, x_3) \in K$  we have that  $a_1 x_1 + a_2 x_2 \leq \sqrt{(a_1)^2 + (a_2)^2} \sqrt{(x_1)^2 + (x_2)^2} \leq a_3 x_3$ , with equality iff  $x \in \mathbb{P}a$ . After some computation we get

$$F_A(x) = \begin{cases} 0 & \text{if } x \in (\text{bd } K) \setminus \mathbb{P}a, \\ x_3/a_3 & \text{if } x \in \mathbb{P}a, \\ \frac{(x_3)^2 - (x_1)^2 - (x_2)^2}{2(a_3 x_3 - a_1 x_1 - a_2 x_2)} & \text{if } x \in \text{int } K. \end{cases}$$

Clearly,  $F_A$  is continuous at  $x \in K$  iff  $x \in K \setminus \mathbb{P}a$ .

(d) Because  $a \in K \setminus \{0\}$ , the set  $I := \{i \in \overline{1, m} \mid \langle a, x_i^* \rangle > 0\} \neq \emptyset$ . Let  $x \in K$  and  $t > 0$ ; then  $x \in t(a + K)$  iff  $x \geq ta$  iff  $t \leq \min \{\langle x, x_i^* \rangle / \langle a, x_i^* \rangle \mid i \in I\}$ . Hence  $F_A(x) = \min \{\langle x, x_i^* \rangle / \langle a, x_i^* \rangle \mid i \in I\}$  for every  $x \in K$ .

Taking into account Theorem 18 (vi) and Example 3 (c) one can ask if for any non polyhedral (pointed closed convex) cone  $K$  one can find  $A \subset K$  such that  $(A, K)$  verify (H) and  $F_A$  be not continuous.

In the sequel we give a positive answer. We begin with the next auxiliary result; the results from Convex analysis used in its proof can be found in [10] or [15].

**Lemma 19** *Let  $C \subset X$  be a closed convex set and  $x_0 \in C$  be such that  $\mathbb{R}_+(C - x_0)$  is not closed. Then there exists a sequence  $(x_n)_{n \geq 1}$  converging to  $x_0$  such that  $x_n \in C \setminus [(1 - \lambda)x_0 + \lambda C]$  for all  $\lambda \in (0, 1)$  and  $n \geq 1$ .*

*Proof.* We may (and do) assume that  $x_0 = 0$ ,  $\dim(\text{lin}(C - x_0)) \geq 2$  and  $\text{int } C \neq \emptyset$ . Indeed, if  $x_0 \neq 0$  we replace  $C$  by  $C - x_0$ ; if  $\dim(\text{lin}(C - x_0)) < 2$  then  $\mathbb{R}_+(C - x_0) = \mathbb{R}_+ C$  is closed; if  $\text{int } C = \emptyset$  we replace  $X$  by  $\text{lin } C$ .

Consider  $u \in [\text{cl}(\mathbb{R}_+ C)] \setminus (\mathbb{R}_+ C)$  with  $\|u\| = 1$  and  $a \in \text{int } C$ ; it follows that  $0 \in \text{bd } C$  and  $sa \in \text{int } C$  for every  $s \in (0, 1]$ . Clearly,  $u$  and  $a$  are linearly independent. There exists a basis  $\{e_1, e_2, \dots, e_p\}$  of  $X$  such that  $e_1 = u$  and  $e_p = a$ . Set  $Y := \text{lin}\{e_1, e_2, \dots, e_{p-1}\}$  and endow  $Y$  with the induced norm; then  $X = Y \oplus \mathbb{R}e_p = Y \oplus \mathbb{R}a$  and  $T : Y \times \mathbb{R} \rightarrow X$ ,  $T(y, s) := y + sa$

is an isomorphism of (normed) linear spaces. Consider the function  $\varphi : Y \rightarrow \overline{\mathbb{R}}$  defined by  $\varphi(y) := \inf\{s \in \mathbb{R} \mid y + sa \in C\}$ . Then  $\varphi$  is convex,  $\varphi(0) = 0$  and  $C \subset T(\text{epi } \varphi)$ . Since  $T(0, 1) = a \in \text{int } C$ , it follows that  $(0, 1) \in \text{int}(\text{epi } \varphi)$ , and so  $\varphi$  is proper and continuous at 0; therefore,  $\varphi$  is Lipschitz on a neighborhood of 0. Because  $C$  is closed, it follows that  $y + \varphi(y)a \in C$  for every  $y \in \text{dom } \varphi$ . Since  $T(0, 1) = a \in \text{int } C$ , there exists  $\varepsilon_0 > 0$  such that  $T(y, 1) \in C$  for every  $y \in Y$  with  $\|y\| \leq \varepsilon_0$ .

(a) For  $t \in (0, \varepsilon_0]$  we have that  $s := \varphi(tu) > 0$ . Indeed, in the contrary case  $s \leq 0$ , and so, for  $\lambda := s/(s-1) \in [0, 1)$ , we get the contradiction  $tu = (1-\lambda)(tu+sa) + \lambda(tu+a) \in C$ .

(b) We have that  $\varphi'(0, u) = 0$ . Indeed, because  $u \in \text{cl}(\mathbb{R}_+ C)$  (and  $C$  is convex with  $0 \in C$ ), there exist the sequences  $(t_n) \subset \mathbb{P}$  and  $(a_n) \subset C$  such that  $a_n := y_n + s_n a \rightarrow 0$  and  $t_n^{-1} a_n \rightarrow u$ . Hence  $y_n \rightarrow 0$ ,  $t_n \rightarrow 0$ ,  $t_n^{-1} s_n \rightarrow 0$  and  $t_n^{-1} y_n \rightarrow u$ . It follows that for  $n$  sufficiently large,  $\varphi(t_n u) \leq \varphi(y_n) + L \|y_n - t_n u\| \leq s_n + L \|y_n - t_n u\|$ , and so  $0 < t_n^{-1} \varphi(t_n u) \leq t_n^{-1} s_n + L \|t_n^{-1} y_n - u\|$  for  $n$  large. Taking the limit we get  $0 = \lim t_n^{-1} \varphi(t_n u) = \varphi'(0, u)$ .

(c) For all  $y \in \mathbb{P}u \cap \text{dom } \varphi$  and  $\lambda \in (0, 1)$  we have that  $\varphi(\lambda y) < \lambda \varphi(y)$ . In the contrary case there exist such  $y$  and  $\lambda$  with  $\lambda \varphi(y) \leq \varphi(\lambda y) = \varphi(\lambda y + (1-\lambda)0) \leq \lambda \varphi(y) + (1-\lambda)\varphi(0) = \lambda \varphi(y) < \infty$ . By the convexity of  $\varphi$  we get  $\varphi(\eta y) = \eta \varphi(y)$  for every  $\eta \in (0, 1)$ , whence  $\varphi'(0, y) = \varphi(y)$ . Since  $y = t_0 u$  with  $t_0 \in \mathbb{P}$ , from (b), we obtain that  $\varphi(tu) = \varphi(y) = \varphi'(0, t_0 u) = 0$  for every  $t \in [0, t_0]$ , contradicting the fact that  $\varphi(tu) > 0$  for  $t \in (0, \varepsilon_0]$  (with  $\varepsilon_0$  from (a)).

Fix now a sequence  $(t_n) \subset (0, \varepsilon_0]$  with  $t_n \rightarrow 0$ , and take  $x_n := t_n u + \varphi(t_n u)a \in C$ . Clearly,  $x_n \rightarrow 0$ . Assuming that  $x_n \in \lambda C$  for some  $\lambda \in (0, 1)$ , we obtain that  $\varphi(\lambda^{-1} t_n u) \leq \lambda^{-1} \varphi(t_n u)$ , and so  $\lambda^{-1} t_n u \in \mathbb{P}u \cap \text{dom } \varphi$ . From (c) we get the contradiction  $\varphi(t_n u) < \lambda \varphi(\lambda^{-1} t_n u) \leq \varphi(t_n u)$ . Therefore,  $x_n \notin \lambda C$  for all  $n \geq 1$  and  $\lambda \in (0, 1)$ .  $\square$

In the next result we complete Theorem 18 vi).

**Corollary 20** *Let  $K \subset X$  be a pointed closed convex cone with nonempty interior. Then  $F_A$  is continuous for every set  $A \subset K$  satisfying condition (H) if and only if  $K$  is polyhedral.*

*Proof.* The sufficiency follows from Theorem 18 vi).

Assume that  $K$  is not polyhedral. Using [13, Prop. 2], there exists  $x_0 \in K$  such that  $\mathbb{R}_+(K - x_0)$  is not closed; of course,  $x_0 \neq 0$ . Let us take  $A := x_0 + K$ . Clearly,  $F_A(x_0) = 1$ . By Lemma 19, there exists a sequence  $(x_n)_{n \geq 1} \subset K \setminus (\frac{1}{2}x_0 + \frac{1}{2}K)$  with  $x_n \rightarrow x_0$ . Because  $x_n \notin \frac{1}{2}A$ , it follows that  $F_A(x_n) \leq \frac{1}{2}$  for every  $n \geq 1$ , and so  $F_A$  is not continuous at  $x_0$ .  $\square$

## 5 Applications to the differentiability of the cost function

In the literature the problem discussed in the previous section is related to the cost function associated to a production function  $F : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$  ( $p \geq 2$ ) satisfying certain conditions; we denote simply  $\geq, \geq, >$  the symbols  $\geq_{\mathbb{R}_+^p}, \geq_{\mathbb{R}_+^p}, >_{\mathbb{R}_+^p}$ , respectively. Among the properties the production function  $F$  could have we mention first those used in [3]:

F.1  $F(0) = 0$ ,

F.2  $F(x) \geq F(x')$  if  $x \geq x'$ ,

F.3  $F$  is quasiconcave,

F.4  $F$  is upper semicontinuous.



In the economics literature some of the conditions above are strengthened:

F.2b  $F(x) > F(x')$  if  $x \geq x'$ ,

F.3b  $F$  is strictly quasiconcave,

F.4b  $F$  is continuous.

The properties of  $F_A$  (defined in (17)) mentioned in Theorem 18 suggest the consideration of the following new conditions:

F.2c  $F(x) > F(x')$  if  $x \geq x'$  and  $F(x') > 0$ ,

F.2d  $F(x) > F(x')$  if  $x \geq x' \in \mathbb{R}_{++}^p$ ,

F.3c  $F$  is strictly quasiconcave on  $\{x \in \mathbb{R}_+^p \mid F(x) > 0\}$ ,

F.3d  $F$  is strictly quasiconcave on  $\mathbb{R}_{++}^p$ ,

F.4c  $F$  is continuous on  $\{x \in \mathbb{R}_+^p \mid F(x) > 0\}$ ,

F.4d  $F$  is continuous on  $\mathbb{R}_{++}^p$ ,

F.5  $F(x) > 0$  if  $x \in \mathbb{R}_{++}^p$ .

Note that in the context of the differentiability of the cost functions Sakai [12] used conditions F.1, F.2, F.4b and the fact that  $F$  is strictly concave instead of F.3b; Avriel *et. al.* [1] used conditions F.2, F.3, F.4; Saijo [11] used conditions F.2b and F.4; Fuchs-Selinger [6] used the condition F.2b.

As in [3], set

$$L(\gamma) := \{x \in \mathbb{R}_+^p \mid F(x) \geq \gamma\} \quad (\gamma \in \mathbb{R}_+).$$

It is well known that:

F.2  $\iff L(\gamma) + \mathbb{R}_+^p \subset L(\gamma)$  for every  $\gamma \in \mathbb{R}_+$ ;

F.3  $\iff L(\gamma)$  is convex for every  $\gamma \in \mathbb{R}_+$ ;

F.4  $\iff L(\gamma)$  is closed for every  $\gamma \in \mathbb{R}_+$ .

In the next proposition we establish several relations among the conditions mentioned above.

**Proposition 21** *Let  $F : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$ .*

- (i)  $F.2b \Rightarrow F.2c \Rightarrow F.2$ ,  $F.2b \Rightarrow F.2d$ ,  $(F.2c \wedge F.5) \Rightarrow F.2d$ ,  $(F.2d \wedge F.4b) \Rightarrow F.2$ .
- (ii)  $F.3b \Rightarrow F.3c \Rightarrow F.3$ ,  $F.3b \Rightarrow F.3d$ ,  $(F.3c \wedge F.5) \Rightarrow F.3d$ ,  $(F.3d \wedge F.4b) \Rightarrow F.3$ .
- (iii)  $(F.2 \wedge F.3z) \Rightarrow F.2z$ ,  $z$  being  $b$ ,  $c$  or  $d$ .
- (iv)  $F.4b \Rightarrow (F.4c \wedge F.4d \wedge F.4)$ ,  $(F.4c \wedge F.4) \Rightarrow F.4b$ ,  $(F.4c \wedge F.5) \Rightarrow F.4d$ .

Proof. (i) Excepting the last one, the implications are obvious.

(F.2d  $\wedge$  F.4b)  $\Rightarrow$  F.2: Let  $x \geq x'$ ; set  $k := x - x' \geq 0$ . There exists  $(x'_n) \subset \mathbb{R}_{++}^p$  such that  $x'_n \rightarrow x'$ ; then  $x_n := x'_n + k \in \mathbb{R}_{++}^p$  and  $x_n \rightarrow x$ . Because  $x_n \geq x'_n$  we have, by F.2d, that  $F(x_n) > F(x'_n)$  for every  $n$ , and so  $F(x) \geq F(x')$  by F.4b.

(ii) Excepting the last one, the implications are obvious.

(F.3d  $\wedge$  F.4b)  $\Rightarrow$  F.3: Let  $x, x' \in \mathbb{R}_+^p$  and  $\lambda \in (0, 1)$ . There exist the sequences  $(x_n), (x'_n) \subset \mathbb{R}_{++}^p$  such that  $x_n \rightarrow x$  and  $x'_n \rightarrow x'$ . Then  $F(\lambda x_n + (1 - \lambda)x'_n) \geq \min\{F(x_n), F(x'_n)\}$  (by F.3d), and so, taking the limit, we get  $F(\lambda x + (1 - \lambda)x') \geq \min\{F(x), F(x')\}$  (by F.4b).



(iii) (F.2  $\wedge$  F.3z)  $\Rightarrow$  F.2z: Let  $x' \geq x$ . Hence  $F(x') \geq F(x)$  by F.2; moreover,  $F(x') > 0$  if  $F(x) > 0$ , respectively  $x' \in \mathbb{R}_{++}^p$  if  $x \in \mathbb{R}_{++}^p$ . Then  $x \neq x'' := 2x' - x \in \mathbb{R}_+^p$  and  $x' = \frac{1}{2}x'' + \frac{1}{2}x$ ; because  $x'' \geq x$  we have  $F(x'') \geq F(x)$  by F.2, and  $F(x') > F(x)$  by F.3z.

(iv) The implications are obvious.  $\square$

When referring to results in the previous sections, in the sequel  $X$  is  $\mathbb{R}^p$  endowed with the Euclidean norm and identified with its dual. Because the conditions F.1, F.2 and F.4 seems to be very natural, in the sequel we also assume that  $F$  verifies these conditions. In this situation, taking  $K := \mathbb{R}_+^p$  and  $A := L(\gamma)$ , we have that  $A = \text{cl } A = A + K \subset K$ , and  $0 \in A$  iff  $\gamma = 0$ . Hence, if  $A \neq \emptyset$  and  $\gamma > 0$  then  $K$  and  $A$  verify condition (H). Set  $\Gamma_F := \{\gamma \in \mathbb{P} \mid L(\gamma) \neq \emptyset\}$ ; clearly  $\text{Im } F \setminus \{0\} \subset \Gamma_F \subset (0, \sup \text{Im } F]$ . If  $F$  is continuous then  $(0, \sup \text{Im } F) \subset \text{Im } F \setminus \{0\}$ .

**Proposition 22** *Let  $F : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$  be continuous and set  $K := \mathbb{R}_+^p$ . Then*

- (a) *F.2c  $\Rightarrow$  [(11) with  $A := L(\gamma)$ ] for every  $\gamma \in \Gamma_F$ ,*
- (b) *F.3c  $\Rightarrow$  [(13) with  $A := L(\gamma)$ ] for every  $\gamma \in \Gamma_F$ .*

Proof. (a) Take  $\gamma \in \Gamma_F$ ,  $x \in L(\gamma)$  and  $k \in K \setminus \{0\}$ ; hence  $F(x) > 0$ . By F.2b we have that  $F(x') > F(x) \geq \gamma$ , where  $x' := x + k$ . Because  $F$  is continuous at  $x'$ , there exists a neighborhood  $V$  of  $x'$  such that  $F(u) \geq \gamma$  for every  $u \in K \cap V$ . Therefore,  $K \cap V \subset L(\gamma)$ , which proves that  $x' \in \text{int}_K L(\gamma)$ . Hence (11) holds.

(b) Take  $\gamma \in \Gamma_F$ ,  $x, x' \in L(\gamma)$  with  $x \neq x'$  and  $\lambda \in (0, 1)$ ; assume that  $F(x) \geq F(x')$ . Since  $F$  is strictly quasiconcave on  $B := \{u \mid F(u) > 0\}$ , we have that  $F(x'') > F(x') \geq \gamma$ , where  $x'' := \lambda x + (1 - \lambda)x'$ . Since  $F$  is continuous on  $B$ , there exists a neighborhood  $V$  of  $x'$  such that  $F(u) \geq \gamma$  for every  $u \in K \cap V$ . Therefore,  $K \cap V \subset L(\gamma)$ , which proves that  $x'' \in \text{int}_K L(\gamma)$ . Hence (13) holds.  $\square$

The following questions are quite natural: Are the converse implications in Proposition 22 true? More precisely, if  $F : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$  is continuous, is it true that F.2c holds if  $L(\gamma)$  satisfies (11) for every  $\gamma \in \Gamma_F$ ? Is it true that F.3c holds if  $L(\gamma)$  satisfies (13) for every  $\gamma \in \Gamma_F$ ?

The answer is negative for both questions. For this take  $G : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$  satisfying conditions F.1, F.2b, F.3b, F.4b and  $\sup G > 1$ ;  $G$  could be defined by  $G(x_1, x_2) := x_1 + x_2 + \sqrt{x_1 x_2}$  for  $(x_1, x_2) \in \mathbb{R}_+^2$ . Take also  $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $\varphi(t) := \min\{t, \max\{1, t - 1\}\}$  and  $F := \varphi \circ G$ ; because  $\varphi$  is a continuous non decreasing function with  $\varphi(0) = 0$ ,  $F$  verifies F.1, F.2b, F.3 and F.4. Then for every  $\gamma \in \mathbb{R}_+$  we have that  $L_F(\gamma) = L_G(\gamma)$  for  $\gamma \in (0, 1]$  and  $L_F(\gamma) = L_G(\gamma + 1)$  for  $\gamma \in (1, \infty)$ . Hence  $L_F(\gamma)$  satisfies conditions (11) and (13) for every  $\gamma \geq 0$  with  $L_F(\gamma) \neq \emptyset$ . Since  $F$  is constant on the nonempty open set  $\{(x, y) \in \mathbb{R}_{++}^p \mid 1 < G(x, y) < 2\}$ , we obtain that  $F$  satisfies neither F.2c nor F.3c.

The next example shows that we can not replace the continuity of  $F$  by its upper semi-continuity in Proposition 22 (b).

**Example 4** ([1, Ex. 4.4]) Let  $F : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$  be defined by

$$F(x_1, x_2) := \begin{cases} x_1 x_2 & \text{if } x_1 x_2 < 1 \text{ or } [\frac{1}{2} < x_1 < 2 \text{ and } x_2 = 1/x_1], \\ 1 + x_1 x_2 & \text{if } x_1 x_2 \geq 1 \text{ and } x_1 + x_2 \geq \frac{5}{2}, \\ 1 + \frac{x_1 x_2 - 1}{(5/2)x_1 - 1 - (x_1)^2} & \text{if } x_1 x_2 > 1 \text{ and } x_1 + x_2 < \frac{5}{2}. \end{cases}$$

The function  $F$  is usc, it is strictly quasiconcave on  $\mathbb{R}_{++}^2 = \{(x_1, x_2) \mid F(x_1, x_2) > 0\}$ , but  $F$  is not continuous [for example,  $F$  is not continuous at  $(2, \frac{1}{2})$ ]. Hence F.1, F.3c and

F.4 hold, but F.4c does not hold. Moreover, F.2c holds by Proposition 21 (iii). However,  $L(5/2) = \{(x_1, x_2) \in \mathbb{R}_+^2 \mid x_1 x_2 \geq 1, x_1 + x_2 \geq \frac{5}{2}\}$  does not verify (13) [or, equivalently, (15)] because  $L(5/2) \subset \mathbb{R}_{++}^2$  and  $\sigma_{L(5/2)}$  is not differentiable on  $-\mathbb{R}_{++}^2$ .

**Corollary 23** *Assume that  $F : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$  is continuous.*

- (a) *If  $F$  satisfies F.3c then  $\sigma_{L(\gamma)}$  is differentiable on  $-\mathbb{R}_{++}^p$  for every  $\gamma \in \Gamma_F$ .*
- (b) *If  $F$  satisfies F.2c and  $\sigma_{L(\gamma)}$  is differentiable on  $-\mathbb{R}_{++}^p$  for every  $\gamma \in \Gamma_F$ , then  $F$  satisfies F.3.*

*Proof.* We know already that  $L(\gamma)$  verifies condition (H) for every  $\gamma \in \Gamma_F$ .

(a) Set  $K := \mathbb{R}_+^p$ . Let  $\gamma \in \Gamma_F$ ; since  $F$  is continuous and satisfies F.3c, we have that  $A := L(\gamma)$  is closed and convex. By Proposition 22 we have that  $A$  verifies condition (13). Applying Proposition 16 we obtain that  $\sigma_{L(\gamma)}$  is differentiable on  $-\mathbb{R}_{++}^p$ .

(b) Take  $\gamma \in \Gamma_F$ . Since F.2c holds, by Proposition 22 we have that condition (11) holds for  $A := L(\gamma)$ . Because  $\sigma_A$  is differentiable on  $-\mathbb{R}_{++}^p$ , using Theorem 15 we obtain that  $A$  is convex. Hence  $L(\gamma)$  is convex for every  $\gamma \in \Gamma_F$ . Since  $L(0) = \mathbb{R}_+^p$ ,  $F$  is quasi-concave, that is, F.3 holds.  $\square$

Having  $F : \mathbb{R}_+^p \rightarrow \mathbb{R}_+$  a production function, the cost function is defined for the price  $x^* \in \mathbb{R}_{++}^p$  and the output  $\gamma \in \mathbb{R}_+$  by

$$c(x^*, \gamma) := \inf \{ \langle x, x^* \rangle \mid x \in L(\gamma) \} = -\sigma_{L(\gamma)}(-x^*).$$

So, the differentiability of  $c(\cdot, \gamma)$  at  $x^* \in \mathbb{R}_{++}^p$  (resp. on  $\mathbb{R}_{++}^p$ ) is equivalent to the differentiability of  $\sigma_{L(\gamma)}$  at  $-x^* \in -\mathbb{R}_{++}^p$  (resp. on  $-\mathbb{R}_{++}^p$ ).

**Remark 2** Sakai [12, Lem. 1 (2)] obtained the differentiability of the cost function for  $F$  strictly concave and satisfying F.1, F.2, F.4b; Saijo [11] (see also Kim [8, Cor. 2]) stated the same result for  $F$  satisfying F.2b, F.3b and F.4, but, as seen in Example 4, this result is not true; Fuchs-Selinger [6, Thm. 2] obtained the differentiability of the cost function for  $F$  satisfying F.1, F.2b, F.3b and F.4b, mentioning, using a figure, that the result of Saijo is not true. Avriel *et. al.* [1, Thm. 4.8] obtained the differentiability of the cost function for  $F$  satisfying F.1, F.2, F.3b and F.4b; our result in Corollary 23 is slightly more general.

**Example 5** Let the set  $A \subset \mathbb{R}_+^3$  (considered in [16]) be defined by

$$A := \left\{ (x, y, z) \in \mathbb{R}_+^3 \mid x + y \leq 1, z \geq \frac{1}{2} \frac{(x + y - 1)^2}{2 - (x - y)^2} \right\} \cup \left\{ (x, y, z) \in \mathbb{R}_+^3 \mid x + y \geq 1 \right\}. \quad (20)$$

The set  $A$  is closed and convex,  $0 \notin A = A + K$ ,  $\sigma_A$  is differentiable on  $\text{int } K^-$ , but (7) does not hold (see [16, Prop. 10]). Moreover,

$$\begin{aligned} \text{int}_{\mathbb{R}_+^p} A &= \left\{ (x, y, z) \in \mathbb{R}_+^3 \mid x + y \leq 1, z > g(x, y) \right\} \cup \left\{ (x, y, z) \in \mathbb{R}_+^3 \mid x + y > 1 \right\}, \\ A + (\mathbb{R}_+^p \setminus \{0\}) &= \text{int}_{\mathbb{R}_+^p} A, \quad \text{int } A = \mathbb{R}_{++}^p \cap \text{int}_{\mathbb{R}_+^p} A, \quad \mathbb{P}A = \mathbb{R}_+^p \setminus \{0\}. \end{aligned}$$

It follows that (11) holds. Since  $a := (1, 0, 0) \in A$ ,  $a' := (0, 1, 0) \in A$  and  $\frac{1}{2}a + \frac{1}{2}a' \notin \text{int}_{\mathbb{R}_+^p} A$ , we have that (13) does not hold. Applying Theorem 18 we have that the corresponding function  $F_A$  is a quasiconcave continuous production function which verifies condition F.2c (even F.2b), but does not verify the condition F.3c (that is  $F_A$  is not strictly quasiconcave on  $\mathbb{P}A$ ); consequently,  $F_A$  is not strictly quasiconcave.

**Remark 3** Corollary 23 (a) confirms the “only if” part of [6, Thm. 2], while Corollary 23 (b), Example 5 and [16, Prop. 10] show that the “if” parts of [6, Thm. 2] and of the Equivalence Theorem in [11] are not true; this is because  $F_A$  is continuous, satisfies F.2b and the corresponding cost functions are differentiable on  $-\mathbb{R}_{++}^3$ , but  $F_A$  is not strictly quasi-concave.

Note that [11, Lem. 3] follows from Proposition 9. Indeed, because (USC) holds and  $y \in \text{Im } F$ ,  $A := L(y) \subset \mathbb{R}_+^p$  is nonempty and closed; moreover,  $\overline{\text{conv}} A \subset \mathbb{R}_+^p$ , and so  $P := (\overline{\text{conv}} A)_\infty \subset \mathbb{R}_+^p$ ; hence  $\mathbb{R}_+^p \subset P^+$ , and so  $\mathbb{R}_{++}^p \subset \text{int } P^+ = \text{int}(\text{dom } g(\cdot, y))$ . In particular, we provided a new proof for Shephard’s Lemma.

From Proposition 9 we get also [5, Thm. 1]; here  $A$  is the set  $R(x) := \{y \in X \mid y \succeq x\}$ ,  $\succeq$  being a reflexive relation on the nonempty closed set  $X \subset \mathbb{R}_+^p$  such that  $R(u)$  is closed for every  $u \in X$  (that is,  $\succeq$  is upper semicontinuous). Because any nonempty closed subset of  $\mathbb{R}_+^p$  can be represented as  $R(x)$  for some upper semicontinuous reflexive relation  $\succeq$  on  $\mathbb{R}_+^p$ , [5, Thm. 2] is very close to Proposition 9 in the case the set  $A$  is a subset of  $\mathbb{R}_+^p$  (because  $(\overline{\text{conv}} A)_\infty \subset \mathbb{R}_+^p$ ).

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